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Rebalancing Strategies for Long-Term Investors

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Abstract
Leading pension plans use asset and liability management systems to optimizise their strategic decisions. The multi-stage models link asset allocation decisions with payments to beneficiaries, changes to plan policies and related issues, to maximize the plan's surplus within a given risk tolerance. Temporal aspects complicate the problem but give rise to special opportunities for dynamic investment strategies. Within these models, the portfolio must be re-revised in the face of transaction and market impact costs. The rebalancing problem is posed as a generalized network with side conditions. We develop a specialized algorithm for solving the resulting problem. A real-world pension example illustrates the concepts.

Keywords: Asset and liability management, financial optimization, multi-stage investment models.

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1. Introduction

Pension plans and other long-term investors can benefit by applying dynamic investment strategies in a consistent fashion over extended time horizons. These investors have several advantages: First, they defer taxes on gains and as a consequence are able to adjust their asset mix in response to changing economic conditions without the burden of tax impacts. Second, they are able to take on greater short-term risks to attain greater long-term returns. In a sense to be shown later, long-term investors are able to capitalize on the inherent volatility often associated with higher-return assets.

Economic growth theory suggests that a multi-stage investor should maximize the expected log of wealth at each time period ([18]). For a given set of scenarios, $s \in S$, we optimize the investment decision as follows:

$$\text{Maximize } \mathrm{E}\,\mathrm{U}(w) = \sum_{s} \pi_s \times \log(w^s)$$

where $\mathrm{U}(w)$ is a von Neumann Morgenstern utility function, log (•)
π_s = probability of scenario $s \in S$, and
w^s = investor's wealth under scenario $s \in S$.
The approach depends upon a series of assumptions, including no transaction and market impact costs, inter-temporal independence of asset returns, and the lack of liabilities and other intermediate cash flows (see [13, 21]). When these assumptions are valid, the multi-stage stochastic optimization problem can be replaced by a series of single-stage, myopic optimization models. The log utility function can be readily replaced by other V-M utility functions, such as iso-elastic family. Section 4 depicts a model possessing a multi-period mean/variance objective function. The resulting sequence of single-stage models is much easier to solve than the corresponding large multi-stage stochastic program.

Alternatively, we can impose a set of policy rules or constraints on the investment process. (The next section provides further details.) Policy rules are generally inspired by solutions to well-known stochastic control problems, such as the longstanding investment/consumption problem ([19, 20, 28]). Herein, for example, the investor rebalances her portfolio to a target mix and consumes a fixed fraction of wealth at each period.

Numerous studies have included transaction costs[1] within a stochastic control context ([5, 7, 10, 12, 16, 29, 30]). These costs are caused by several factors, including taxes, thin or volatile markets, and large trades. Generally, the results can be described in terms of an allowable zone around the target asset proportion—called a no-trade-zone.

To illustrate the advantages of rebalancing a portfolio to a target benchmark at each time period, we turn to historical data from the years 1972 to 1997. The asset universe consists of four prominent asset categories:

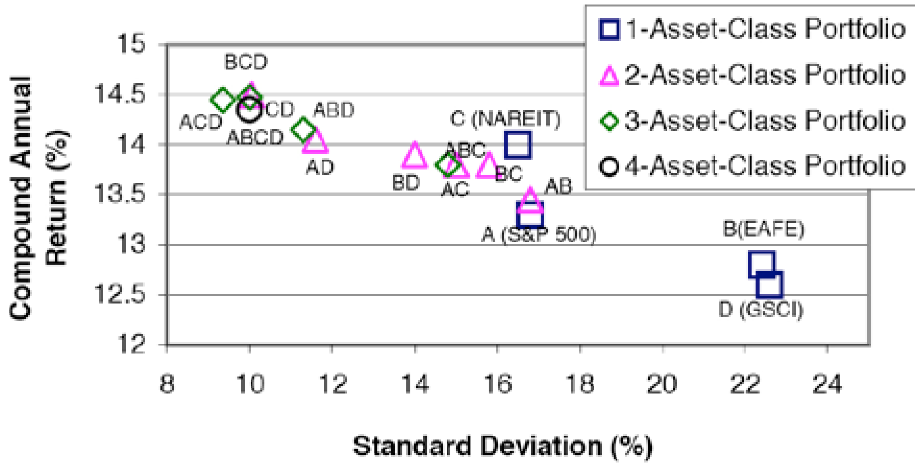
- S&P 500 (large US companies),
- EAFE (foreign stocks from Europe, Australia, and Far East),
- REIT (real estate investment trusts), and
- GSCI (Goldman Sachs commodity index).

The historical asset returns are plotted in mean/risk space (Figure 2.1), indicating compound returns and risk via standard deviation of return over the 26-year historical period. In addition, we show the accompanying returns and risk for a set of dynamically balanced portfolios. These return/risk characteristics dominate any individual asset category. In fact, this type of relationship is common over extended periods, for a wide variety of markets and alternative time periods. Multi-asset portfolio performance will be even better when individual asset categories possess

1 - We include all fees and market impact costs in the term–transaction costs.

greater volatility and are relatively uncorrelated. This supplemental return is due to the volatility occuring in markets. See [11] for a theoretical discussion of volatility pumping. Of course, the portfolio must be actively rebalanced at the beginning of each time period.

Figure 2.1. Reward/risk Characteristics for several fixed-mix strategies (historical analysis: 1972-1997)



The next section defines the multi-stage optimization model for investing in assets over an extended time horizon. This model is able to address real-world issues, such as transaction costs, in a comprehensive fashion. In Section 3, we show that the rebalancing problem can be posed as a generalized network with side constraints. This result is new and it leads to the empirical evidence discussed in Section 4.

2. Multi-Period Investment Model

This section defines the pension plan problem as a multi-stage stochastic program. The basic model is a variant of Mulvey *et al.* [24], with special attention to transaction costs.

To define themodel, we divide the entire planning horizon T into two discrete time intervals T_1 and T_2, where $T_1 = \{0, 1,..., \tau\}$ $\alpha\nu\delta$ $T_2 = \{\tau + 1,..., T\}$. The former corresponds to periods in which investment decisions are made. Period τ defines the date of the planning horizon; we focus on the investor's position at the beginning of period τ. Decisions occur at the beginning of each time stage. Much flexibility exists. An active trader might see his time interval as short as minutes, whereas a pension plan advisor will be more concerned with much longer planning periods such as the dates between the annual Board of Director's meeting. It is possible for the step-sizes to vary over time–short intervals at the beginning of the planning period and longer intervals towards the end. T_2 handles the horizon at time τ by calculating economic and other factors beyond period τ up to period T. The investor renders passive decisions after the end of period τ.

Asset investment categories are defined by set A = $\{1, 2, ..., I\}$,with category 1 representing cash. The remaining categories can include broad investment groupings such as growth and value stocks, bonds, and real estate. The categories should track well-defined market segments. Ideally, the co-movements of pairs of asset returns would be relatively low so that diversification can be done across the asset categories.

As with single-period models, uncertainty is represented by a set of distinct realizations $s \in S$. Scenarios may reveal identical values for the uncertain quantities up to a certain period—*i.e.*, they share common information history up to that period. Scenarios that share common information must yield the same decisions up to that period. We address the representation of the information structure through non-anticipativity conditions. These constraints require that any variables sharing a common history, up to time period t, must be set equal to each other. See equation (2.8).

We assume that the portfolio is actively rebalanced at the beginning of each period. Alternatively, we could simply make no transaction except to reinvest any dividend and interest—a buy-and-hold strategy. For convenience, we also assume that the cash flows are reinvested in the generating asset category and all the borrowing (if any) is done on a single period basis.

For each $i \in A$, $t \in T_1$, and $s \in S$, we define the following parameters and decision variables.

Parameters:

$r^s_{i,t} \quad = 1+\rho^s_{i,t}$ where $\rho^s_{i,t}$ is the percent return for asset i, time period t, under scenario s (projected by a stochastic scenario generator, for example, see [25]).

π_s Probability that scenario s occurs, $\sum_{s=1}^{S} \pi_s = 1$

w_0 Wealth at the beginning of time period 0.
$\sigma_{i,t}$ Transaction costs incurred in rebalancing asset i at the beginning of time period t (symmetric transaction costs are assumed, *i.e.*, cost of selling equals cost of buying).
β^s_t Borrowing rate in period t, under scenario s.

Decision variables:

$x^s_{i,t}$ Amount of money in asset category i, at the beginning of time period t, under scenario s, after rebalancing.
$v^s_{i,t}$ Amount of money in asset category i, at the beginning of time period t, under scenario s, before rebalancing.
w^s_t Wealth at the beginning of time period t, under scenario s.
$p^s_{i,t}$ Amount of asset i purchased for rebalancing in period t, under scenario s.
$d^s_{i,t}$ Amount of asset i sold for rebalancing in period t, under scenario s.
b^s_t Amount borrowed in period t, under scenario s.

Given these definitions,we present the deterministic equivalent of the stochastic asset-only allocation problem.

Model (SP)

$$\text{Max} \quad \text{E U}(w_\tau) = \sum_{s=1}^{S} \pi_s \text{U}(w^s_\tau) \tag{2.1}$$

s.t.

$$\sum_i x^s_{i,0} = w_0 \qquad \forall s \in S \tag{2.2}$$

$$\sum_i x^s_{i,\tau} = w^s_\tau \qquad \forall s \in S \tag{2.3}$$

$$v^s_{i,t} = r^s_{i,t-1} x^s_{i,t-1} \qquad \forall s \in S,\ i \in A,\ t = 1,\dots,\tau \tag{2.4}$$

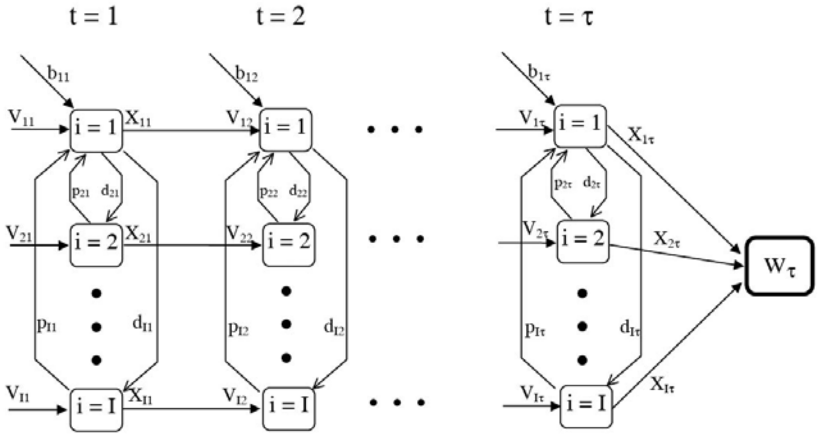
$$x^s_{i,t} = v^s_{i,t} + p^s_{i,t}(1-\sigma_{i,t}) - d^s_{i,t} \qquad \forall s \in S,\ i \neq 1,\ t = 1,\dots,\tau \tag{2.5}$$

$$x^s_{1,t} = v^s_{1,t} + \sum_{i \neq 1} d^s_{i,t}(1-\sigma_{i,t}) - \sum_{i \neq 1} p^s_{i,t} - b^s_{t-1}(1+\beta^s_{t-1}) + b^s_t \qquad \forall s \in S,\ t = 1,\dots,\tau \tag{2.6}$$

$$x_{i,t}^{s} \geq 0 \qquad \forall s \in S,\, i \in A,\, t = 1, \ldots, \tau \tag{2.7}$$

$$x_{i,t}^{s} = x_{i,t}^{s'} \qquad \forall s, s' \in S \text{ with identical past up to time } t. \tag{2.8}$$

Figure 2.2. Network representation for each scenario $s \in S$.



A generalized network investment model is presented in Figure 2.2. This graph depicts the flows across time for each of the asset categories. The transaction costs are shown in equations (2.5) and (2.6) as linear or piecewise linear functions of the amount of trading over the respective arc. While all constraints cannot be put into a network model, the graphical form is easy for managers to comprehend. General linear and nonlinear programs, the preferred model, are now readily available for solving the resulting problem. However, a network may have computational advantages for extremely large problems, such as security level models. See [8] for applications of generalized networks.

As with single-period models, the nonlinear objective function (2.1) can take several different forms. If the classical return-risk function is used, (2.1) becomes Max $Z = \eta$ Mean (w_τ)- $(1\text{-}\eta)$ Risk (w_τ), where Mean (w_τ) is the expected total wealth and Risk (w_τ) is the risk of the total wealth across the scenarios at the end of period τ. Parameter η indicates the relative importance of risk as compared with the expected value. This objective leads to an efficient frontier of wealth at period η by allowing alternative values of η in the range [0,1]. As noted above, a viable alternative to mean-risk is the von Neumann-Morgenstern expected utility of wealth at period τ.

Constraint (2.2) guarantees that the total initial investment equals the initial wealth. Constraint (2.3) represents the total wealth in the beginning of period τ. This constraint can be modified to include assets, liabilities, and investment goals, in which case the modified result is called the surplus wealth ([21]). Many investors render investment decisions without reference to their liabilities or investment goals. Mulvey ([21]) uses the notion of surplus wealth to the mean-variance and the expected utility models to address liabilities in the context of asset allocation strategies. Constraint (2.4) depicts the wealth $v_{i,t}^{s}$ accumulated at the beginning of period t before rebalancing in asset i. The flow balance constraint for all assets except cash for all periods is given by constraint (2.5). This constraint guarantees that the amount invested in period t equals the net wealth for the asset. Constraint (2.6) represents the flow balancing constraint for cash. Non-anticipativity constraints are represented by (2.8). These constraints ensure that the scenarioswith the same pastwill have identical decisions up to that period. Although these constraints are numerous, solution algorithms take advantage of their simple structure ([2, 6, 15, 22]).

Model (SP) depicts a split variable formulation of the stochastic asset allocation problem. This formulation has proven successful for solving the model using techniques such as the progressive hedging algorithm of Rockafellar and Wets and the DQA algorithm by Mulvey and Ruszczynski ([22]). The split variable formulation can be beneficial for direct solvers that use the interior point method.

By substituting constraint (2.8) back in constraints (2.2) to (2.6), we obtain a standard form of the stochastic allocation problem. Constraints for this formulation exhibit a dual block diagonal structure for two stage stochastic programs and a nested structure for general multi-stage problems. This formulation may be better for some direct solvers. The standard form of the stochastic program possesses fewer decision variables than the split variable model and is the preferred structure by many researchers in the field. This model can be solved by means of decomposition methods, for example, the L-shaped method (a specialization of Benders algorithm). See [2, 6, 9].

The multi-stage model can provide superior performance over single period models. See the references [6, 9, 25, 31].

We generalize the asset-only investment model to address pension plan issues. Mostly, pension plan administrators must make periodic cash (or stock) contributions and pay beneficiaries to the plan's retirees. We define two sets of decision variables:

e_t^s Cash contributions at time t, scenario s.
p_t^s Payment to beneficiaries at time t, scenario s.

A pension plan must conduct annual evaluations to determine the plan's ability to pay its beneficiaries in the future. To this end, actuaries calculate the plans surplus or deficit as follows:

$$Sw_t^s = w_t^s - \text{Present value}\left(p_{t+1}^s, p_{t+2}^s, \ldots, p_{\tau+2}^s\right) \tag{2.9}$$

where the present value is taken over the nodes in the subtree emanating out of the node (s, t).

Generally, a contribution payment is required when the plan falls into deficit or when there are obligations from previous time periods. The exact contribution depends on actuarial rules and the structure under which the plan operates. These rules are complex formulae based on the company's position and the existing economic environment. We define these relationships with the simple functional form as follows:

$$e_t^s = f(Sw_t^s) \tag{2.10}$$

To complete the model, we add the cash inflows (contribution) and cash outflows (payments to beneficiaries) to the cash balance equation (2.6) at each time period and under each scenario. The result is a financial planning system for a pension plan.

$$\begin{gathered} x_{1,t}^s = v_{1,t}^s + \sum_{i \neq 1} d_{i,t}^s (1 - \sigma_{i,t}) - \sum_{i \neq 1} p_{i,t}^s - b_{t-1}^s (1 + \beta_{t-1}^s) + b_t^s + e_t^s - p_t^s \\ \forall s \in S, t = 1, \ldots, \tau \end{gathered} \tag{2.11}$$

To simplify (SP), we develop a model possessing a special policy rule, called fixed mix or dynamically balanced, as a special case of (SP). Define the proportion of wealth to be: $\lambda_{j,t}^s$ for each asset $j \in A$, time period $t \in T$, under scenario $s \in S$. A dynamically balanced portfolio enforces the following condition at each juncture:

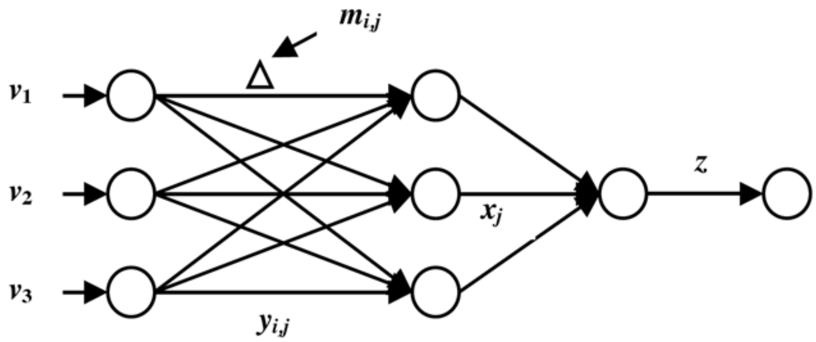
$$\lambda_j = \frac{x_{j,t}^s}{w_t^s}, \text{ where } \lambda_j = \lambda_{j,t}^s \tag{2.12}$$

This constraint ensures that the fraction of wealth in each asset category $j \in A$ is equal to λ_j at the beginning of every time period. Ideally, we would maintain the target λ fractions at all time periods and under every scenario. Practical considerations prevent this simple rule from being implemented in a direct fashion. Rather, we define a no-trade zone for modeling the rebalancing decision. The goal of this approach is to minimize trading within the no-trade zone and to rebalance the portfolio whenever the asset proportions fall outside their respective zones. The next section develops an optimization model for achieving these goals. Adding decision rules to model **(SP)** gives rise to a non-convex optimization model. Thus, the search for the best solution requires specialized algorithms.

3. The Portfolio Revision Problem

This section describes a method for revising a portfolio at each stage during the multi-stage investment model. First, we concentrate on the constraints imposed by the lower and upper bounds on the no-trade-zones, in the context of a network model. Figure 2.3 shows a generalized network graph for rebalancing the portfolio at each time period. (The time and scenario indices are dropped to simplify the presentation).

Figure 2.3. Generalized network model for revising portfolio (at time period *t*)



We define two sets over the asset categories, $i \in A$ and $j \in A$, representing asset values before and after rebalancing, respectively, at each time period t. The variable $y_{i,j} \geq 0$ depicts the transfer of funds from asset i to asset j in order to achieve the required no-trade-zone restrictions. In concert, we define multiplier $m_{i,j}$ for the accompanying transaction costs on arc (i, j).

Flow conservation constraints for the proposed rebalancing problem are shown below.

$$\sum_j y_{i,j} = v_i \qquad \forall i \in A \tag{2.13}$$

$$\sum_i m_{i,j} y_{i,j} = x_j \qquad \forall j \in A \tag{2.14}$$

The total wealth after rebalancing and paying transaction and market-impact costs equals z, as shown below. The objective function is to maximize the flow variable z, while ensuring that assets remain within the no-trade zones for the assets after the portfolio is rebalanced.

$$\sum_j x_j = z \tag{2.15}$$

The no-trade-zone rule is employed when rebalancing the portfolio in the face of transaction costs. Rather than always rebalancing the portfolio to the λ_j targets, we add a zone around each target:

No-trade-zone for asset $j = [\lambda_j - l_j - \lambda_j + u_j]$ for parameters l_j and u_j for each asset category.

Whenever an asset falls outside the no-trade zone, the portfolio is rebalanced so that all asset proportions fall within their respective zones. If possible, only assets outside the no-trade zones are bought or sold; however, in certain cases, assets within the zones must be traded in order to enforce the overall rule. To fulfill the no-trade-zone restrictions, we ensure that the x variables remain within the associated boundaries for each variable:

$$l_j \times z \leq x_j \leq u_j \times z \qquad \forall j \in A \tag{2.16}$$

We propose:

Model (B)

Maximize z

Subject to equations (2.12) to (2.16)

for rebalancing a portfolio at the beginning of each time period. This model ensures that the assets lie within their respective no-trade zones, while striving to minimize overall transaction costs, *i.e.* to maximize the z-variable.

Rather than solving (B) directly[2], we propose an auxiliary model in which the lower and upper restrictions on the x-variables (equations 2.16) are replaced by the following constraint:

Model (B-1)

$$l_j \times \hat{z} \leq x_j \leq u_j \times \hat{z} \qquad \forall j \in A \tag{2.17}$$

where $\hat{z}$ is an estimate for wealth after transaction costs, and

$$z = \hat{z} \tag{2.18}$$

The $\hat{z}$ estimate should be set in a manner so as to fix the relationship between models (B) and (B-1). For instance, we could find the lowest transaction costs to bring infeasible assets back to feasibility. Then, the value $\hat{z}$ is set equal to the current wealth minus the lower bound on transaction costs. Relationships between the two models are described next.

We propose a search algorithm for finding a feasible solution with the lowest amount of transaction costs, i.e. the largest value of z. The idea is to find the maximum flow through the proposed generalized network. As a complication, the bound constraints on the x-variables do not fit the generalized network and must be addressed as side conditions.

Theorem 1. Any feasible solution to model **(B-1)** is a feasible solution to the original problem **(B)**.

Proof:

If model **(B-1)** has a feasible solution, the solution to **(B-1)** fulfills the constraints 2.13 by construction. Thus, it also fulfills the remaining constraints for **(B)**.

Corollary 1. An infeasible solution to **(B-1)** is infeasible to **(B)**.

The corollary follows directly since the side conditions are enforced within model **(B-1)**. To determine a convergent algorithm, we rely on the following theorem.

Theorem 2. The set of feasible solution values of **(B-1)**, as a function of $\hat{z}$ forms a convex set.

Proof:

Model (B) is a linear program. Hence, its objective function value over the feasible region defines a convex set. Models **(B)** and **(B-1)** are equivalent in terms shown in Theorem 1; hence the set of feasible solution values of **(B-1)**, as a function of $\hat{z}$ forms a convex set.

As a consequence, the solution to **(B)** is equivalent to the maximum of the $\hat{z}$ values over the values of $\hat{z}$ that give rise to feasible solutions to **(B-1)**.

Corollary 2. The optimal solution to original problem is equivalent to the maximum of model **(B-1)** over feasible $\hat{z}$ values.

The proposed solution algorithm is developed as a function of these relationships. First, we estimate the smallest amount of transaction costs to pay in order to rebalance the portfolio and set $\hat{z}$ equal to wealth at the end of the previous period minus the estimated costs. Next, we apply a maximum

2 - See (8) for a discussion of generalized network applications.

flow algorithm over a generalized network to solve model (**B-1**). If a feasible solution is found, the optimal solution has been located. Otherwise, we lower the estimate of $\hat{z}$ and re-solve the new (**B-1**) model. A bisection, or related search, method can be used to find the optimal solution to the desired accuracy.

In the next section, we implement a procedure similar to the one developed herein within a policy-optimization model for a large US pension plan.

4. Pension Plan Example

In this section, we describe results with optimizing a real-world portfolio for a large US pension plan. As discussed in the previous section, the dynamically balanced approach is taken in concert with selected no-trade zones to reduce transaction costs while capitalizing on volatility pumping. The problem involves the strategic asset allocation decisions for a large US pension plan.The company divides its assets into eleven categories as shown in Table 2.1.

The problem studied here is a multi-stage portfolio optimization problem. For each asset class, mean return and standard deviation of return are calculated from historical data. For any pair of asset classes, covariance is also calculated from historical data. A ten-year planning horizon is studied. It is assumed that portfolio rebalancing is done quarterly. Thus, the problem is a multi-stage portfolio optimization over forty periods. The goal is to maximize the risk and rewards for total asset wealth (and surplus) at the end of the fortieth period. One thousand scenarios are generated for each asset class using a systemof stochastic differential equations. Based on these scenarios, the mix that maximizes the final total wealth is found. Transactions and market impact costs are included in the analysis, as described in Sections 2 and 3. The projected cost values represent a combination of brokerage fees, market impact costs, and related costs.[3]

Next, we highlight the results of the multi-stage study.[4] A multi-stage approach provides information that is difficult to gather in a single period approach. Significant issues such as rebalancing rules and temporal tradeoffs can be evaluated. Carrying out an asset-liability management study requires a set of capital market assumptions. Table 2.1 presents the relevant information for the 10-year planning period. These projections combine historical values (*e.g.* volatility), with judgmental factors (*e.g.* equity risk premiums) and current market conditions (*e.g.* long bond rates). In addition, a covariance matrix was generated in a similar manner, mostly with historical values for the correlations. Due to policy rules, several asset categories were given upper limits, for example, private equity at 8%. In addition, the no-trade zones are displayed, along with estimated market impact costs.

Given this data, it is straightforward to generate a single-period Markowitz efficient frontier for the assets (Figure 2.4). The lower-risk side of the frontier contains low volatility assets (*e.g.* cash and bonds), while the upper-risk side contains mostly equity. Remember that the top of the frontier is a linear program; thus, the highest returning asset is emphasized to the highest degree possible. There is little or no diversification, unless the investor puts upper bounds on higher returning assets. The client's current position is in the moderate risk area. The highest returns lie between 8.5% and 9% due to policy constraints on equity assets.

As the next step, we developed amulti-stage scenario generator to model the underlying economic factors, accompanied by the corresponding returns for the eleven assets. The factors consisted of interest rates, inflation, economic activity, and corporate earnings in the US. A global model was deemed inappropriate because of the nature of the company. The calibration targets for asset returns are identical to the single-period model. In addition, we required temporal series for the underlying factors so that the liabilities could be generated in a consistent fashion with the assets.

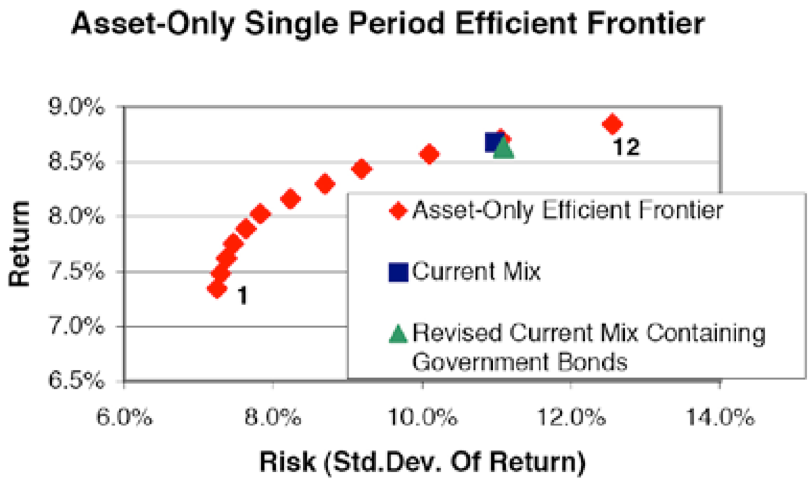
3 - For example, it is difficult to change asset proportions for private equity and hedge funds, due to the requirements for long-term commitments.
4 - This example is patterned after a real world pension plan, but the numbers have been modified and identities disguised.

Table 2.1. Capital market assumptions for pension plan example

Asset Category	Expected Return	Volatility	Target	Lower Bounds	Upper Bounds	Transaction Costs (%)
Cash	5.0%	1.0%	0.050	0	0.10	0
S&P 500 Equity	9.0%	16.0%	0.260	0.20	0.47	0.011
Private Equity	14.0%	24.5%	0.040	0.03	0.08	0.150
European Equity	9.0%	18.0%	0.075	0.03	0.13	0.025
Far East Equity	9.0%	18.0%	0.075	0.03	0.13	0.025
Emerging Market Eq.	9.0%	27.0%	0.080	0.02	0.09	0.150
Corp/Gov't Bonds[a]	7.0%	6.0%	0.040	0.02	0.06	0.005
International Bonds	6.8%	7.0%	0.080	0.06	0.14	0.007
High-yield bonds	8.3%	8.0%	0.220	0.20	0.25	0.075
Real Estate	7.8%	14.0%	0.070	0.05	0.10	0.100
Hedge Funds	8.3%	7.0%	0.010	0	0.05	0.100

[a]Salomon BIG index.

Figure 2.4. Single period efficient frontier



Based on these scenarios, we solved a non-convex optimization model for determining the best asset mixes over the ten-year planning horizon. The asset efficient frontier at the end of the planning horizon provides recommendations that are different from the single-period model. First, the top of the efficient frontier has a mixture of equity assets, rather than the typical single asset coming out of a single-period optimizer. The optimal solution in terms of returns encompasses a range of equity assets, including S&P500, European and Far East, private equity, and emergingmarket equity. The investor improves performance by diversifying and rebalancing at each period to a target proportion.

Also, the returns are higher—almost 9.6% at the maximum. Higher returns are due to the rebalancing gains attributed to the fixed-mix policy rule.[5] The multi-period model diversifies the portfolio to a much greater degree than the single-period model, especially for investor with a moderate to high tolerance for risk.[6]

Once a multi-stage scenario generator/decision simulator is constructed, the investor can readily evaluate a number of issues. To give an example, we test the impact of adding contributed stock from two subsidiaries (Table 2.2). The contributed stocks increase the plan's surplus and will be sold over the next three to five years in a linear fashion and replaced by S&P 500 stock to improve the plan's diversification. The goal is to increase the company's surplus position so that it can better withstand difficult economic conditions in the future. Not only does the contributed stock improve the pension surplus, but it also enhances returns at the higher ends of the frontier (Figure 2.5), despite the fact that the contributed stocks possess the usual 9% equity expected returns

5 - We re-balanced the portfolio each quarter.
6 - Long-term investors should be able to take on higher short-term risks, especially if they can withstand reasonable volatility induced by recessions, short-term crises and related events.

and will be sold within three to five years. Greater returns are due to rebalancing gains; the contributed stocks' volatility (39% and 44%) gives an advantage to long-term investors. See [11, 18] for an explanation of rebalancing gains and the role of volatile assets in multi-stage models.

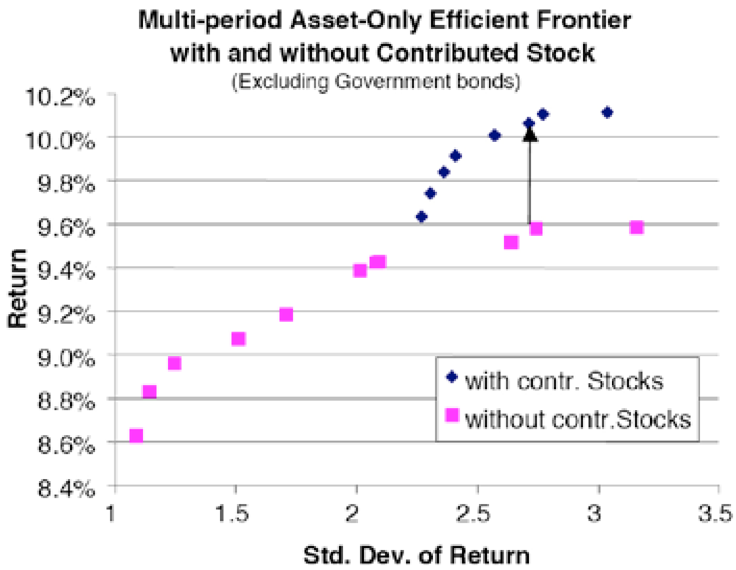
Table 2.2. Additional asset categories

Asset Category	Expected Return	Volatility	Transaction Costs
Contributed stock 1	9%	44%	0.100
Contributed stock 2	9%	39%	0.100
US government bonds	6.75%	10%	0.075

The recommendations derived from a ten-year efficient frontier must be tempered by the analysis of shorter-term impacts. Can the company tolerate the year-to-year uncertainties? There are many ways to evaluate this issue. For instance, we are interested in the plan's contributions and expenses over time.

These distributions are easily computed from the ALM system.

Figure 2.5. Multi-period asset efficient frontier (returns are higher than single-period model due to rebalancing gains).



We show the optimal asset solution from the standpoint of a surplus analysis. Here, we simulate the actuarial and accounting requirements for paying beneficiaries, for making contributions, and paying PBGC (Pension Benefit Guaranty Corporation) expenses. The company's actuaries supplied the data so that we could estimate these values for each scenario over the ten-year planning horizon. The full pension plan simulator can then be optimized in order to locate sets of dominating solutions. As mentioned, the actual problem consists of multidimensions; it cannot be simplified to a single or even a small number of graphs. Nevertheless, we will depict two surplus efficient frontiers in order to emphasize an important point—namely, the suitability of asset categories should be fixed in concert with their relationship to the liabilities. To illustrate, we add a long-government bond asset category to the analysis. This asset more closely links to the economic changes in the liabilities than any of the other assets. Thus, it gives greater protection during economic downturns in which interest rates drop (Figure 2.6) than other bond categories. In addition, the long-bond asset can improve returns over longer periods of time as a result of its volatility.[7] The ten-year surplus efficient frontier (Figure 2.7) also improves with the addition of government long-bonds. Again, the multi-stage model gives insights into the nature of long-term investment strategies.

7 - The expected returns from long bonds are slightly less than the Salomon BIG, but have greater volatility. Thus, long-bonds take advantage of volatility pumping over long time horizons.

Figure 2.6. Surplus efficient frontier at end of year one (government bonds provide cushion against economic downturn).

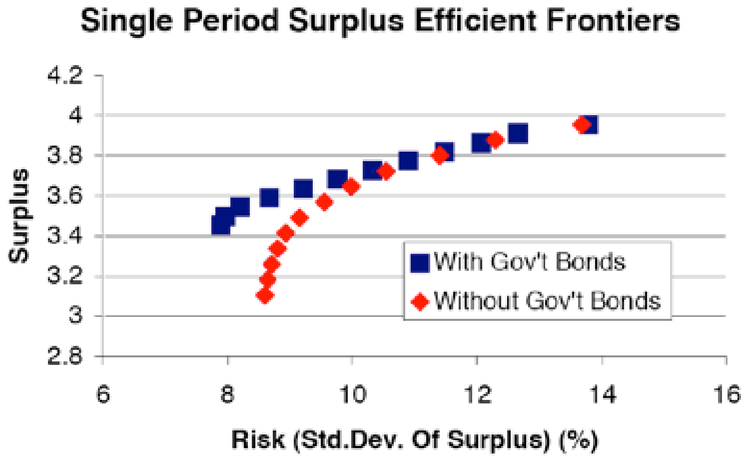
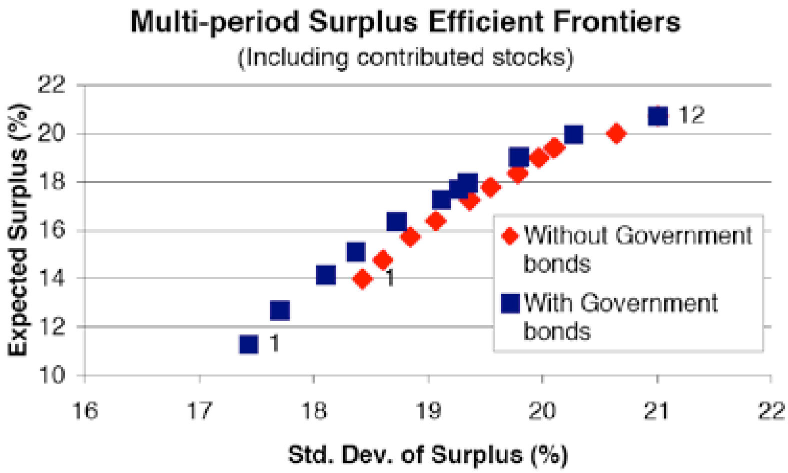


Figure 2.7. Surplus efficient frontier at end of 10th Year (government bonds take advantage of volatility pumping).



5. Conclusions

The report has shown that investment performance can be improved by means of dynamically rebalancing a portfolio, as compared with the recommendation of a single-period mean/variance model. The results are noteworthy, especially for less risk adverse investors. No-trade zones assist in this regard by reducing the total amount of transaction costs. The generalized network provides a practical method for implementing these concepts. In addition, we illustrated that volatility pumping improves portfolio performance while at the same time reducing portfolio risks. The multi-asset approach for dynamic investing has several advantages, but the investor must address transaction, market impact costs and other real world concerns.

What are directions for future research? First, we could evaluate alternative methods for taking advantage of dynamic re-balancing rules, while minimizing total transaction costs. The lower and upper bounds can be determined by means of decision variables. There is a clear tradeoff between portfolio return and total transaction costs. A stochastic programming model may be helpful in this regard.

A related research topic involves the comparison of policy rules (such as the dynamically balanced rule discussed herein) and the multi-stage stochastic programming framework. There are pros and cons to each of the approaches. Likely, a synthesis may prove the best in the long run. As an example, stochastic programming solutions might inspire better policy rules than the dynamically balanced rule, especially when coupled with non-parametric statistical estimates. Also, stochastic programs can blend tactical and strategic plans in a single seamless planning system.

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