

Fairness in Trading – A Microeconomic Interpretation



Stephen Satchell

Professor of Finance, Birkbeck College & University of Cambridge

Bernhard Scherer

Professor of Finance, EDHEC Business School

Abstract

We show that non-linear transaction costs generate external effects between accounts due to trade volume dependent marginal transaction costs. For an asset manager with multiple clients this raises the question of fairness. How do I ensure I treat all clients fairly? In general, two possible solutions exist. The first is the so-called Cournot/Nash solution, where each account is optimized under the assumption that trading in the remaining accounts is given. However, in a Cournot/Nash equilibrium each client pays the average costs of trading but creates higher marginal costs (under the assumption of non-linear transaction costs) on the "community" of accounts. Ignoring this interdependence will hurt performance in all accounts. We model optimal trading with mean variance preferences as a duopoly game. This allows us to use well developed microeconomic tools for analysing the optimal trading problem and link it with the literature on external effects and their solution, *i.e.*, the COASE theorem.

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1. Introduction

Optimal trading and portfolio construction in the presence of transaction costs have received widespread attention during the last decade.¹ However, all analysis has been performed in the presence of a single representative account. In practice, asset management companies have many accounts (of varying size) that engage in trading the same assets. It has long been overlooked that non-linear transaction costs are creating an externality from one account to another.² Trading one account raises the marginal transaction costs for the next account and the current practice of sequentially optimizing accounts generally fails to take this into account. As a consequence all accounts will be worse off. In general, two possible solutions exist. The first is the so-called Cournot/Nash solution, where each account is optimized under the assumption that trading in the remaining accounts is given. However, in a Cournot/Nash equilibrium each client pays the average costs of trading but creates higher marginal costs (under the assumption of non-linear transaction costs) on the "community" of accounts. Ignoring this interdependence will hurt performance in all accounts. We will argue that we can make all clients better off in a collusive equilibrium where all accounts are jointly optimized. This will render the Cournot/Nash equilibrium unacceptable as an asset manager that is required to achieve the best for his clients (see FSA Principle #6: "A firm must pay due regard to its customers and treat them fairly"). This paper will show that separate optimizations need to be combined and enhanced to reflect the interactions between accounts due to trading. Given this aggregation of trading requirements, an optimization problem must be formulated that will allow each client to "see" the cost of trading based on the aggregate trading volume and not just on his own volume. The conundrum here is that we cannot determine one client's trading needs until we know those of all the others. How do we get this process started? The answer is a simultaneous optimization across all accounts.

We start in section 2 with a more formal yet still generic description of accounting for interdependencies between accounts in portfolio construction. Section 3 interprets the results of section 2 in a standard microeconomic setting. We model the optimal trading with mean variance preferences as a duopoly game. This allows us to use well developed microeconomic tools for analysing the optimal trading problem and link it with the literature on external effects and their solution, *i.e.*, the COASE theorem. Section 4 provides some numerical insights and section 5 concludes with an application to finding optimal capacity in an investment product.

2. Competitive Cournot/Nash Solution versus PARETO Optimality

The following section provides a generic description of the issues that arise if an asset management firm manages multiple accounts. We model the value added to a client account, $U_j(\cdot)$, defined as utility³ after transaction costs. For the j -th (out of m) accounts this amounts to

$$(1) \quad U_j \left(a_j, \sum_{i=1}^m a_i \right) = u(a_j) - \tau \left(\sum_{i=1}^m a_i \right) \frac{a_j}{\sum_{i=1}^m a_i}$$

where a_j denotes the amount of trading (asset weight times account size) in a single asset, a , for account j and $\tau(\cdot)$ describes total transaction costs as a function of total trading that is split proportionally, $\frac{a_j}{\sum_{i=1}^m a_i}$, across accounts. In other words we see that value added is a function of trading in account $j(a_j)$, as well as the trading that occurs in total over all accounts, $\sum_{i=1}^m a_i$.

While the former directly affects utility *before* transaction costs, $u(a_j)$, the latter affects total transaction costs $\tau = \tau \left(\sum_{i=1}^m a_i \right)$. Our transaction cost function is assumed to be convex such that marginal costs exceed average costs

$$(2) \quad \frac{d\tau \left(\sum_{i=1}^m a_i \right)}{da_j} > \frac{\tau \left(\sum_{i=1}^m a_i \right)}{\sum_{i=1}^m a_i}$$

1 - For a review see Kissel/Glantz (2003).

2 - O'Connide/Scherer/Xu (2006) have so far been the only source of intuition on this issue.

3 - Note that utility is also the most natural and impossible to game risk-adjusted performance measure available to economists.

If the transaction cost function was linear, all accounts could be optimized separately, as trading in one account would not affect marginal transaction costs (constant). Trading in any given account would not affect the optimal level of trading in another account.

We will first look at the competitive solution, *i.e.*, each account trades with a liquidity provider (the market) taking the amount of trading in the other account as given. This will lead to a Cournot/Nash equilibrium. The first-order condition for (1) in this setting (where we optimize the j -th account while keeping all other accounts unchanged) is simply found by taking the first-order derivative in (1) with respect to a_j .

$$(3) \quad \frac{dU(n_j, \sum n_i)}{dn_j} = \frac{du(a_j)}{da_j} - \frac{d\tau(\sum a_i)}{da_j} \cdot \frac{a_j}{\sum a_i} + \frac{\tau(\sum a_i)}{\sum a_i} - \frac{a_j \tau(\sum a_i)}{(\sum a_i)^2} = 0.$$

Rearranging (3) we see that the optimal solution trades off marginal utility from investment performance versus transaction costs which in turn are a trading-weighted combination of marginal and average transaction cost

$$(4) \quad \frac{du(n_j)}{dn_j} = \left[\frac{n_j}{\sum n_i} \right] \cdot \frac{d\tau(\sum n_i)}{dn_j} + \left(1 - \left[\frac{n_j}{\sum n_i} \right] \right) \cdot \frac{\tau(\sum n_i)}{\sum n_i} \cdot$$

From here we can also calculate the external effect on the value added of account j if we start trading slightly less for another account, k .

$$(5) \quad \frac{dU(n_j, \sum n_i)}{dn_k} = - \left[\frac{n_j}{\sum n_i} \right] \left[\frac{d\tau(\sum n_i)}{dn_k} - \frac{\tau(\sum n_i)}{\sum n_i} \right] < 0$$

The externality is always negative, because (by assumption) marginal costs exceed average costs. So while nobody has an incentive to deviate in the above NASH equilibrium (ensured by the first-order condition $\frac{dU(n_j, \sum n_i)}{dn_j} = 0$), the accounts could do collectively better by trading a little less as $\frac{dU(n_j, \sum n_i)}{dn_k} < 0$. The NASH solution is not PARETO optimal and can improved upon. A joint optimization across all accounts

$$(6) \quad U = \sum_{j=1}^m U_j \left(n_j, \sum_{i=1}^m n_i \right)$$

is clearly preferable, as the combined utility function will directly manage the interaction effect (5). The external affect from trading is nest internalized by running a joint optimization. In other words, the COASE theorem would predict that asset managers that are following similar strategies ($a_1 = a_2 = \dots = a_m$) should merge to improve performance after transaction costs. Not merging would leave them with the (transaction) costly NASH equilibrium.

3. Differences in Value Added – A Duopoly Game

3.1 Model Setup

In this section we want to become more specific by providing closed form solutions for the difference in value added as well as some numerical illustrations of the above concept. We assume two accounts with different size (assets under management) s_i that choose asset weights n_i .⁴ The transaction cost function is quadratic and takes the form

$$(7) \quad \tau = \frac{\theta}{2} (n_1 s_1 + n_2 s_2)^2$$

i.e., each account trades a volume that is equal to account size times asset weight, $s_i n_i$, starting

4 - In the notation of the previous section this means that $a_i = s_i n_i$.

from a zero position. The transaction costs borne by each account will be calculated by splitting up total trading costs according to the relative trade size starting from a zero position. The transaction costs borne by each account will be calculated by splitting up total trading costs according to the relative trade size

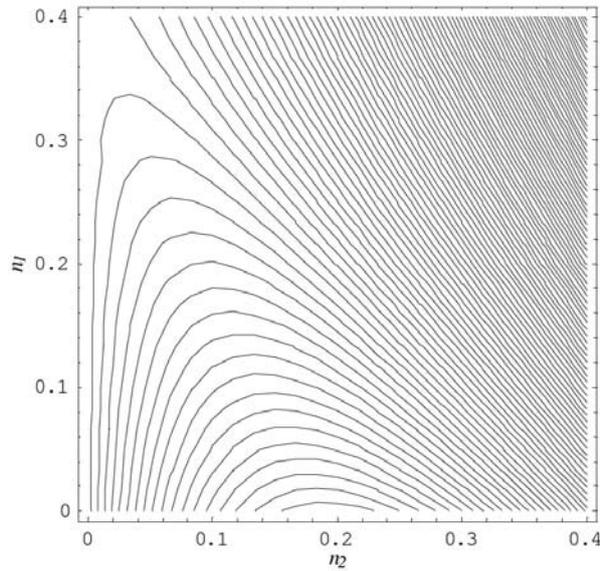
$$(8) \quad \tau(\cdot) = \tau(\cdot) \cdot \frac{(n_1 s_1)}{(n_1 s_1 + n_2 s_2)} + \tau(\cdot) \cdot \frac{(n_2 s_2)}{(n_1 s_1 + n_2 s_2)}$$

while the value added for each account becomes

$$(9) \quad VA_i = n_i s_i \mu - \frac{\lambda}{2} (n_i s_i)^2 \sigma^2 - \frac{\theta}{2} (n_i s_i + n_j s_j) (n_i s_i)$$

In (9) we assumed standard mean variance preferences (utility function). Expected (excess) return and risk on the risky asset are given by μ and σ^2 respectively, while risk aversion is captured by λ . In (9) we need to "leverage" up the otherwise "dimensionless" return and risk measures used in portfolio optimization as the last expression in (9) expresses the total trading costs allocated to account i . We can illustrate the objective function (9) for a given account. Assume the following parameters: $s_1 = s_2 = 1, \mu = 0.1, \sigma = 0.2, \lambda = 0.5, \theta = 0.5$. If we plot, for example, VA_2 for all possible combinations of n_1 and n_2 as a contour-plot we arrive at Figure 1.

Figure 1. ISO-Value added curve for account number 2. Each curve represents a different level of value added (utility after transaction costs). The level of value added falls as we move away from the n_2 axis.



The optimal solution for account 2 is reached on the n_2 axis, where account 1 trades nothing. This is where VA_2 is maximal. The farther we move from this axis, the more value added is lost to account 2 as we reach more and more distant ISO-value added curves. Each point with zero slope on those curves represents the optimal trading for account 2 given an amount of trading for account 1. Moving marginally to the left or right (trading marginally less or more) would leave us with a more distant ISO-value-added curve and hence lower utility after subtraction of transaction costs.

3.2 The Illusory Stand-Alone Solution

As a reference point, we start deriving the so-called stand-alone solution. In other words, what would be the optimal solution to an account, if it was treated as if it were the only account? This reflects current asset management practice in which every account is constructed separately. It equally applies to quantitative portfolio construction processes where each account is independently run through a portfolio optimizer. Of course, this solution is illusory.

The optimal solution is trivially given by

$$(10) \quad n_i^{SA} = \arg \max_{n_i} \left[n_i s_i \mu - \frac{\lambda}{2} (n_i s_i)^2 \sigma^2 - \frac{\theta}{2} (n_i s_i)^2 \right] = \frac{\mu}{s_i (\theta + \lambda \sigma^2)}$$

Note that the objective function differs from (9) as any interdependencies are ignored. The second account does not exist in the mind of the portfolio engineer. Substituting this back into the objective function, we would get

$$(11) \quad VA_i^* = \frac{\mu^2}{2(\theta + \lambda \sigma^2)}$$

if no other account had been traded. While this is the first best solution, it is elusive, as it ignores any interaction between accounts. Of course, it will turn out that this solution is very disadvantageous if external effects from one account to the other exist. In reality, we need to substitute (10) into (9) instead to arrive at

$$(12) \quad VA_i^{SA} = \frac{1}{2} \frac{\lambda \mu^2 \sigma^2}{(\theta + \lambda \sigma^2)^2}$$

$$(13) \quad VA_i^{SA} + VA_j^{SA} = \frac{\lambda \mu^2 \sigma^2}{(\theta + \lambda \sigma^2)^2}$$

for the individual and joint value added. The difference between (11) and (12) will always be positive, *i.e.*, the ignoring the interaction will make you end up overestimating the available value added

$$(14) \quad VA_i^* - VA_i^{SA} = \frac{\mu^2}{2(\theta + \lambda \sigma^2)} - \frac{\lambda \mu^2 \sigma^2}{(\theta + \lambda \sigma^2)^2} = \frac{1}{2} \frac{\theta \mu^2}{(\theta + \lambda \sigma^2)^2} > 0$$

by trading too much. Optimal portfolio construction with multiple accounts is designed to improve collectively on (13) and individually on (12).

3.3 Cournot/Nash Solution

In case we have more than one client, an asset manager could optimize each account separately but for a given amount of trading for the second account. This is otherwise known as the Cournot/Nash solution. Maximum value added VA_i^{CN} for each account is found from the usual first-order conditions on (9).

$$(15) \quad \frac{dVA_i}{dn_i} = \mu s_i - \lambda n_i s_i^2 \sigma^2 - \theta n_i s_i^2 - \frac{1}{2} \theta n_j s_i s_j = 0$$

Solving (15) for n_i we create a so called "reaction function", *i.e.*, the optimal amount of trading in account i for a given amount of trading in account j

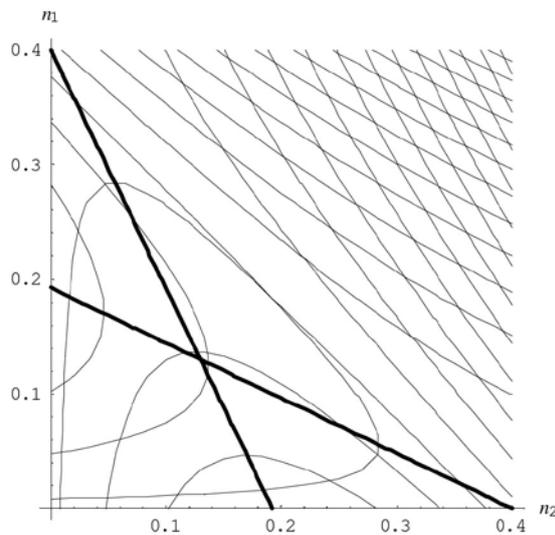
$$(16) \quad n_i = \frac{\mu}{s_i (\theta + \lambda \sigma^2)} - \frac{\theta s_j}{s_i (\theta + \lambda \sigma^2)} n_j$$

$$(17) \quad n_j = \frac{\mu}{s_j (\theta + \lambda \sigma^2)} - \frac{\theta s_i}{s_j (\theta + \lambda \sigma^2)} n_i$$

From (16) we see that the optimal trading for account i depends positively on the "stand-alone solution", $\frac{\mu}{s_i (\theta + \lambda \sigma^2)}$, as well as negatively on the interaction effect between both accounts. This effect will increase with size and trading in account in the second account.

In order to illustrate the solution with a numerical example we choose as parameterization $s_1 = s_2 = 1, \mu = 0.1, \sigma = 0.2, \lambda = 0.5, \theta = 0.5$ to arrive at the following visualization in Figure 2 below. Both reaction functions are plotted as negatively sloped thick lines while the contour plot expresses the value added after transaction costs for both accounts. The maximum value added rises as we get closer to the respective axis. For example: the maximum value added for account 2 rises if account 2 invests 19.23% in the risky asset while account 1 invests nothing. If account 1 also starts to trade, the value added for account 2 decreases while the value added for account 1 increased (it reaches ISO-value added contours closer to the n_1 axis). The equilibrium is reached where both reaction functions cross. The reaction functions cut through the maximum on each respective ISO-value added curve. This is not surprising given their derivation. They represent the optimal trading (maximum value added) for a given trading of the other account. Even slightly different trading will, by definition, no longer be optimal.

Figure 2: Reaction function for Cournot-Nash solution. The reaction functions for n_1 and n_{12} become $n_1 = 0.4 - 2.082n_2$ and $n_2 = 0.192308 - 0.480769n_1$, for the above parameterization of $s_1 = s_2 = 1, \mu = 0.1, \sigma = 0.2, \lambda = 0.5, \theta = 0.5$. In equilibrium, both accounts trade 12.987% in the risky asset.



We can derive this point by substituting (17) in (16) and vice-versa. This allows us to calculate the (symmetric) optimal trade sizes for both accounts.

$$(18) \quad n_j^{CN} = \frac{2\mu}{s_j(3\theta + 2\lambda\sigma^2)}$$

$$(19) \quad n_i^{CN} = \frac{2\mu}{s_i(3\theta + 2\lambda\sigma^2)}$$

In our numerical example this amounts to an equilibrium trading of 12.99% in the risky asset. We can now substitute (18) and (19) into (9) to arrive at individual as well as collective value added.

$$(20) \quad VA_i^{CN} = \frac{2\mu^2(\theta + \lambda\sigma^2)}{(3\theta + 2\lambda\sigma^2)^2}$$

$$(21) \quad VA_i^{CN} + VA_j^{CN} = \frac{4\mu^2(\theta + \lambda\sigma^2)}{(3\theta + 2\lambda\sigma^2)^2}$$

The Cournot/Nash solution trades less than the stand-alone solution, as we can already see from Figure 2. Equilibrium trading is "to the left" of both stand-alone solutions. Formally, we can see that from

$$(22) \quad n_j^{CN} - n_j^{SA} = \frac{2\mu}{s_j(3\theta+2\lambda\sigma^2)} - \frac{\mu}{s_j(\theta+\lambda\sigma^2)} = -\frac{\theta\mu}{s_j(\theta+\lambda\sigma^2)(3\theta+2\lambda\sigma^2)} < 0$$

This has an immediate effect on the difference in value added

$$VA_i^{CN} - VA_i^{SA} = \frac{2\mu^2(\theta+\lambda\sigma^2)}{(3\theta+2\lambda\sigma^2)^2} - \frac{\lambda\mu^2\sigma^2}{2(\theta+\lambda\sigma^2)^2} = \frac{\mu^2\theta^2(4\theta+3\lambda\sigma^2)}{2(\theta+\lambda\sigma^2)^2(3\theta+2\lambda\sigma^2)^2} > 0$$

which will always be positive (as we assumed $\theta > 0$). Of course, if transaction costs were zero, any interdependence between accounts would have been removed.

3.4 Collusive Solution

In a collusive solution we maximize joint value added instead and therefore arrive at a PARETO optimal solution. The attractiveness of this solution stems from the fact that it is well known that external effects can be best dealt with by internalizing them into the objective function. This is the gist of the so called COASE-Theorem in microeconomics.

$$(23) \quad VA = n_i s_i \mu - \frac{\lambda}{2} n_i^2 s_i^2 \sigma^2 - \frac{\theta}{2} (n_i s_i + n_j s_j) (n_i s_i) + n_j s_j \mu - \frac{\lambda}{2} n_j^2 s_j^2 \sigma^2 - \frac{\theta}{2} (n_i s_i + n_j s_j) (n_j s_j)$$

Our two first order conditions to (23) can be found from

$$(24) \quad \frac{dVA}{dn_i} = \mu s_i - \lambda n_i s_i^2 \sigma^2 - \theta n_i s_i^2 - \theta n_j s_i s_j = 0$$

The difference in (24) to (15) arises from the additional derivative, $\frac{dVA_j}{dn_i} = -\frac{\theta}{2} n_j s_i s_j$. In other words, we now incorporate the external effect on account j from trading account i into our first-order conditions rather than ignoring it as it has been the case with the Cournot/Nash solution. Solving this system of two equations in two unknowns we get

$$(25) \quad n_i^C = \frac{\mu}{s_i(2\theta + \lambda\sigma^2)}$$

$$(26) \quad n_j^C = \frac{\mu}{s_j(2\theta + \lambda\sigma^2)}$$

and after substitution and some tedious algebra⁵

$$(27) \quad VA_i^{CN} = \frac{2\mu^2(\theta + \lambda\sigma^2)}{(3\theta + 2\lambda\sigma^2)^2}$$

$$(28) \quad VA_i^C + VA_j^C = \frac{\mu^2}{2\theta + \lambda\sigma^2}$$

How does the collusive solution compare with the Cournot/Nash, both in terms of trading as well as value added? After all, this was what we have been set to achieve—an improvement of the Cournot/Nash solution.

$$(29) \quad n_j^{CN} - n_j^{SA} = \frac{2\mu}{s_j(3\theta+2\lambda\sigma^2)} - \frac{\mu}{s_j(\theta+\lambda\sigma^2)} = -\frac{\theta\mu}{s_j(\theta+\lambda\sigma^2)(3\theta+2\lambda\sigma^2)} < 0$$

As initially suspected, the Cournot/Nash solution does individually and collectively worse than the PARETO optimal collusive solution. This directly arises from reduced trading, as we can see from

5 - We see that the second-order condition given by $\frac{d^2VA}{dn_i^2} \cdot \frac{d^2VA}{dn_j^2} - \left(\frac{d^2VA}{dn_i dn_j}\right)^2 = \lambda s_i^2 s_j^2 \sigma^2 (2\theta + \lambda\sigma^2) > 0$ for a (local) maximum is satisfied.

$$(30) \quad n_i^C / n_i^{CN} = 1 - \frac{\theta}{4\theta + 2\lambda\sigma^2} < 1$$

The ratio of optimal weights for both solutions will always be smaller than 1. Again, if costs were zero we would trivially arrive at the same solution.

4. Should Size Matter?

In this section we slightly alter the model setup. The mean variance utility function employed so far made actual trading independent of wealth, *i.e.*, the size of the account. In other words all solutions shared the property that the chosen asset weight has been a function of account size in order to keep total trading (weight times account size) constant. This is not necessarily realistic, as asset managers usually choose weights n_i irrespective of account size. Small accounts should look like clones (same weights). However, the wealth independence of mean variance preferences could not recover this result. Without changing our framework too much we can express risk aversion as an inverse function of wealth, *i.e.*,

$$(31) \quad \lambda_i = s_i^{-1}$$

Larger accounts will now show a lower risk aversion and hence we will need to trade more for these accounts to maximize their utility, *i.e.*, risk-adjusted performance. The optimal solution under our collusive approach is now given by

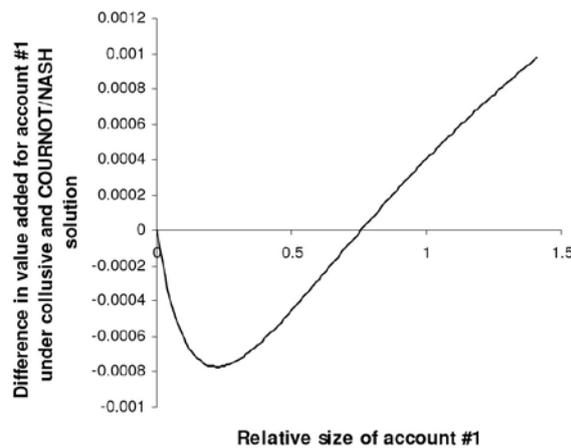
$$(32) \quad n_1^C = n_2^C = \frac{\mu}{\theta(s_1 + s_2) + \sigma^2}.$$

Optimal weights are the same for both accounts, but still dependent on total assets under management ($s_1 + s_2$), as our intuition would suspect.⁶ After repeating the analysis in the previous section we can express the difference in value added between the monopoly and the Cournot/Nash solution:

$$(33) \quad VA_1^C - VA_1^{CN} = \frac{1}{2} \mu^2 s_1 \left[\frac{1}{\theta(s_1 + s_2) + \sigma^2} - \frac{4(\theta s_1 + \sigma^2)(\theta s_2 + 2\sigma^2)^2}{(3\theta^2 s_1 s_2 + 4\theta(s_1 + s_2)\sigma^2 + 4\sigma^4)^2} \right]$$

While it must still be true that *collective* utility is higher in the collusive solution than in the Cournot/Nash solution, from (33) it does not look guaranteed any more that the monopolistic solution will always yield a higher value added for *individual* utility. To get further insight we will illustrate our analysis with a numerical example for the following parameters: $s_2 = 1, \mu = 0.1, \sigma = 0.3, \theta = 0.5$.

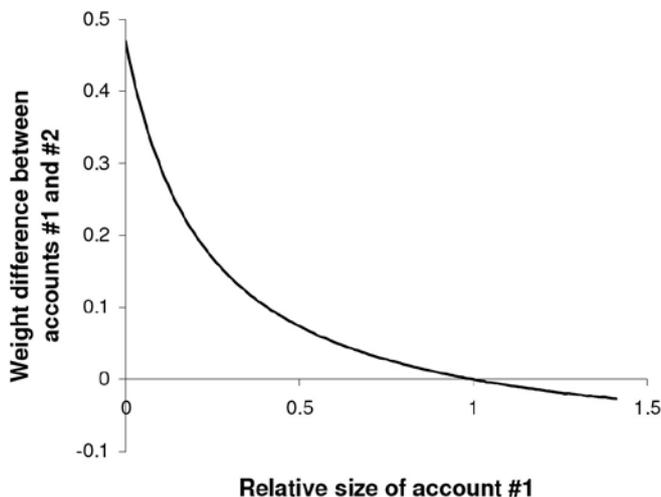
Figure 3. Collusive solution and account size. We plot $VA_1^C - VA_1^{CN}$ versus relative account size. As long as account #1 is small, it will prefer the Cournot/Nash solution to the collusive solution.



6 - We get $VA_i^C = \frac{1}{2} \frac{\mu^2(s_i)}{\theta(s_1 + s_2) + \sigma^2}$ and joint utility is split ($\frac{VA_i^C}{VA_1^C + VA_2^C} = \frac{s_i}{s_1 + s_2}$) proportionally between accounts.

Why does our setup no longer guarantee that $VA_i^C \geq VA_i^{CN}$? From Figure 3 we see that small accounts would benefit from a Cournot/Nash solution as long as it is "small". The intuition here is that as we know that the Cournot/Nash solution trades too much (as it neglects the "feedback" interaction between both accounts) the smaller account will benefit from this excess trading at the expense of the larger account. This can be seen from Figure 4.

Figure 4. Relative weight and account size in Cournot/Nash solution. The smaller account receives a much larger weight allocation under the Cournot/Nash solution.

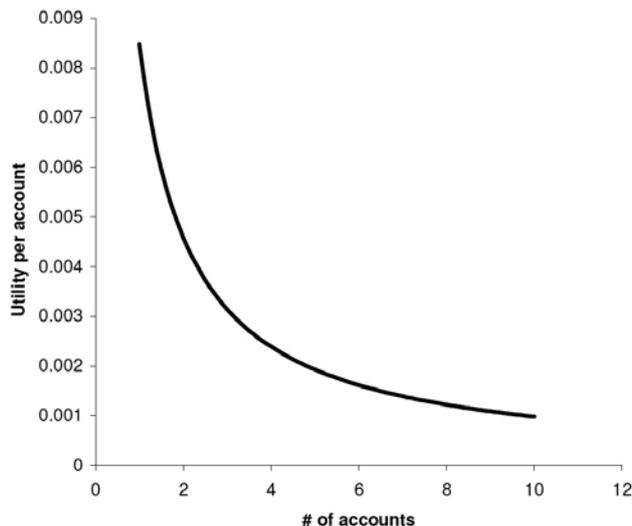


Not only does this seem unfair, as the smaller client will get better performance due to a larger allocation in the positive return asset, but it is also inefficient, as total utility must be smaller than under the collusive solution.

5. Conclusion and Asset Management Implication

We argued that a joint collusive (monopolistic) solution is preferable to a Cournot/Nash solution or a stand-alone approach. Our above model can explain why investment firms with similar investment processes merge into larger units. This is simply an implication of the COASE theorem, *i.e.*, that there is an incentive (better performance across all accounts and therefore larger fees) for a market solution to deal with external effects by internalizing them.

Figure 5. Capacity and assets under management. We plot utility (risk-adjusted performance measure) per account against the number of equal sized accounts.



Very much related to this, we can also address the issue of capacity (how many assets can be managed with a given investment strategy), which is primarily a transaction cost (market impact) issue. From Figure 5 (under the previous parameter assumptions) we can see our framework

applied to the capacity question. Performance per account comes down as the number of accounts increases. It will never become negative, however, as individual portfolio weights would decline. However, for this to take place an asset management firm needs to optimize simultaneously across all accounts. To determine the optimal capacity we just need to set a level of "acceptable utility" (y-axis) and read of the maximum number of accounts that can sustain that performance (x-axis).

6. References

- Coase R., 1960, The Problem of Social Cost, *Journal of Law and Economics* v3 n1, p.1-44.
- Kissel R. and M. Glantz, 2003, *Optimal Trading Strategies*, AMACOM.
- O'Kinneide C., B. Scherer, and X. Xu, 2006, Ensuring Fairness When Pooling Trades, *Journal of Portfolio Management*, v32 n4, Summer, p.33-43.